Int. J. Multiphase Flow Vol. 11, No. 4, pp. 481-502, 1985
Printed in the U.S.A.

0301-9322/85 $3.00 + .00
Pergamon/Elsevier

PROPAGATION OF LONG WAVES OF FINITE AMPLITUDE AT THE INTERFACE OF TWO VISCOUS FLUIDS

G. OOMS, A. SEGAL and S. Y. CHEUNG
Delft University of Technology, Delft, The Netherlands

and

R. V. A. OLIEMANS
Koninklijke/Shell-Laboratorium Amsterdam, Amsterdam, The Netherlands
(Shell Research B.V.)

(*Received* 1 *July* 1984; *in revised form* 30 *January* 1985)

Abstract—The propagation of long waves of finite amplitude at the interface of two viscous fluids has been studied theoretically. For plane Couette-Poiseuille flow of two superposed layers of fluids of different viscosity, an equation is derived to determine the development in time of the shape of these finite amplitude waves. The influence of the viscosity ratio, the density difference of the fluids and an imposed pressure gradient have been investigated.

1. INTRODUCTION

A theoretical model for oil–water core-annular flow through a horizontal pipe was developed by Ooms *et al.* (1984). According to this model, the movement of the wavy oil core with respect to the pipe wall induces pressure variations in the water film, which can exert a force on the core in the vertical direction. This force can be so great that it counterbalances the buoyancy force on the core owing to a density difference between oil and water, allowing a steady core-annular flow to arise. The waves on the interface are essential: If the amplitudes of the waves become zero, there will no longer be a force on the core to counteract the buoyancy force. In such a case, the core will rise or fall in the pipe, until it touches the pipe wall. The magnitude of the force also depends to a large extent on the shape of the waves: When the waves are symmetrical, again, no counteracting force will be present. For an application of the model, knowledge is required about the amplitude and shape of the interfacial waves. In the original model, the oil viscosity was assumed to be so high that any flow in the core, and hence any variation in the oil–water interface form with time, could be neglected. So the core was assumed to be solid and the interface to be a solid–liquid interface. The simplification to a solid core permitted a free choice of the shape and amplitude of the waves; these were chosen in accordance with observations on oil–water core-annular flow experiments.

In reality, the oil core has a finite viscosity, and the amplitude and shape are determined by the gravity force, the pressure drop over the pipe and the surface tension. The amplitude and shape cannot be chosen freely. However, the calculation of the finite amplitude waves for a core-annular flow with an eccentric core through a horizontal pipe is very complicated. Therefore, as a first approximation to the real flow problem, we have calculated the finite amplitude waves for a plane Couette-Poiseuille flow of two superposed layers of fluids of different viscosity between two horizontal plates. As the thickness of the water film is usually very small in comparison with the wavelength and the pipe radius, we believe that this approximation can be rather realistic.

Yih (1967) showed that plane Couette-Poiseuille flow of two superposed layers of fluids of different viscosity between two horizontal plates can be unstable to a long-wavelength perturbation, however small the Reynolds number is. The cause of instability was found to be the difference in viscosity. Similar results were found by Li (1969) for a three-layer viscous stratified flow between two horizontal plates and by Hickox (1971) for the flow of two fluids flowing concentrically in a straight circular tube. Hooper & Boyd (1983) studied the stability of a cocurrent flow of two fluids of different viscosity in an unbounded region; they found that the interface may be unstable to a small-wavelength perturbation.

Yih wondered what will happen with the flow when it is unstable and disturbed slightly. According to him, one certainly cannot expect turbulence to be the final result when the Reynolds number is small. Yih supposed that owing to the growth of the disturbances, long waves of finite amplitude occur at the interface. The purpose of this paper is to investigate if such waves are possible and, if so, to calculate their shapes.

Yih assumes in his paper that the amplitude of the disturbance is very small compared to the distance between the plates. The complete motion can then be resolved into the primary motion and a small perturbation motion. The perturbation motion is described by two Orr-Sommerfeld equations coupled by the interface conditions. Yih's solution procedure for the Orr-Sommerfeld equations and interface conditions is a perturbation calculation with the ratio of the distance between the plates and the disturbance wavelength as the small perturbation parameter. In first approximation, the perturbations are found to be neutrally stable; in second approximation, unstable modes of disturbance are found.

In our calculation, the amplitude of the disturbance is not small compared to the distance between the plates, and so the Orr-Sommerfeld equations cannot be used. We will start with the Navier-Stokes equations. However, like Yih, we will simplify these equations by using a perturbation calculation with the ratio of the distance between the plates and the disturbance wavelength as the small perturbation parameter. Results for the first approximation are given. The influence of the viscosity ratio of the two fluids and the density difference between the fluids and of the imposed pressure gradient on the wave amplitude and shape is shown. To check the results, it will be assumed in some of the calculations that the disturbance amplitude is very small, as for very small amplitudes our results must be in agreement with Yih's first-approximation results.

2. THEORY

The flow problem

Figure 1 gives a sketch of the flow problem. At the interface between fluid 1 and fluid 2, long waves of finite amplitude are supposed to be present. The waves are assumed to be two dimensional. As a reference system, a system is chosen according to which lower plate is at rest. The upper plate has a velocity u_w in the x-direction with respect to the reference system.

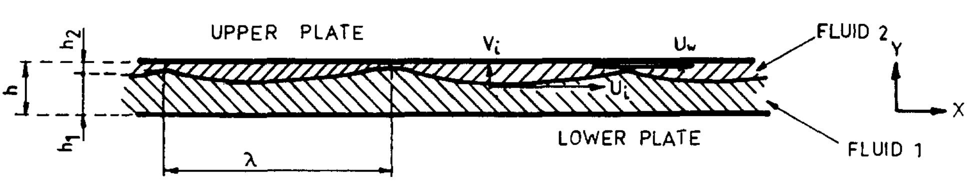

Figure 1. Sketch of the flow problem.

The waves are supposed to be periodic; their wavelength is λ. The velocity of the fluids at the interface is given by the components $u_i(x, t)$ and $v_i(x, t)$. The thickness of fluid layer 1 is given by $h_1(x, t)$ and that of fluid layer 2 by $h_2(x, t)$. Their sum is, of course, constant and equal to h. The fluids are supposed to be incompressible; so their densities ρ_1 and ρ_2 are constant. The viscosities are given by η_1 and η_2.

The flow of the two fluids can be calculated with the aid of the continuity equations

$$\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} = 0, \quad [1]$$

$$\frac{\partial u_2}{\partial x} + \frac{\partial v_2}{\partial y} = 0, \qquad [2]$$

and the Navier-Stokes equations

$$\rho_1 \frac{\partial u_1}{\partial t} + \rho_1 u_1 \frac{\partial u_1}{\partial x} + \rho_1 v_1 \frac{\partial u_1}{\partial y} = -\frac{\partial \phi_1}{\partial x} + \eta_1 \frac{\partial^2 u_1}{\partial x^2} + \eta_1 \frac{\partial^2 u_1}{\partial y^2}, \qquad [3]$$

$$\rho_1 \frac{\partial v_1}{\partial t} + \rho_1 u_1 \frac{\partial v_1}{\partial x} + \rho_1 v_1 \frac{\partial v_1}{\partial y} = -\frac{\partial \phi_1}{\partial y} + \eta_1 \frac{\partial^2 v_1}{\partial x^2} + \eta_1 \frac{\partial^2 v_1}{\partial y^2}, \qquad [4]$$

$$\rho_2 \frac{\partial u_2}{\partial t} + \rho_2 u_2 \frac{\partial u_2}{\partial x} + \rho_2 v_2 \frac{\partial u_2}{\partial y} = -\frac{\partial \phi_2}{\partial x} + \eta_2 \frac{\partial^2 u_2}{\partial x^2} + \eta_2 \frac{\partial^2 u_2}{\partial y^2}, \qquad [5]$$

$$\rho_2 \frac{\partial v_2}{\partial t} + \rho_2 u_2 \frac{\partial v_2}{\partial x} + \rho_2 v_2 \frac{\partial v_2}{\partial y} = -\frac{\partial \phi_2}{\partial y} + \eta_2 \frac{\partial^2 v_2}{\partial x^2} + \eta_2 \frac{\partial^2 v_2}{\partial y^2}, \qquad [6]$$

supplemented with appropriate boundary conditions to be discussed later on. u_1 and v_1 are the velocity components of fluid 1; u_2 and v_2 those of fluid 2. t represents the time; the x- and y-coordinates are chosen as shown in figure 1. The pressure variables ϕ_1 and ϕ_2 are given by

$$\phi_1 = p_1 + \rho_1 g y, \qquad [7]$$

$$\phi_2 = p_2 + \rho_2 g y, \qquad [8]$$

in which p_1 and p_2 are the pressures in the fluids and g is the acceleration due to gravity.

Approximate flow equations

Next, the order of magnitude of the terms in [1]–[6] is determined. As a length scale in the x-direction, λ is used, and as a length scale in the y-direction h. As velocity scale in the x-direction, u_w is used and in the y-direction v. The order of magnitude of the terms in [1] is

$$\underset{O\left(\frac{u_w}{\lambda}\right)}{\frac{\partial u_1}{\partial x}} + \underset{O\left(\frac{v}{h}\right)}{\frac{\partial v_1}{\partial y}} = 0. \qquad [9]$$

This yields

$$v = O\left(u_w \frac{h}{\lambda}\right). \qquad [10]$$

The time scales in the x-direction (λ/u_w) and in the y-direction (h/v) are of the same order of magnitude. With the aid of [10], the order of magnitude of the inertial and viscous terms of [3] and [4] can be written as

$$\underset{O\left(\frac{\rho_1 u_w^2}{\lambda}\right)}{\rho_1 \frac{\partial u_1}{\partial t}} + \underset{O\left(\frac{\rho_1 u_w^2}{\lambda}\right)}{\rho_1 u_1 \frac{\partial u_1}{\partial x}} + \underset{O\left(\frac{\rho_1 u_w^2}{\lambda}\right)}{\rho_1 v_1 \frac{\partial u_1}{\partial y}} = -\frac{\partial \phi_1}{\partial x} + \underset{O\left(\frac{\eta_1 u_w}{h^2} \cdot \frac{h^2}{\lambda^2}\right)}{\eta_1 \frac{\partial^2 u_1}{\partial x^2}} + \underset{O\left(\frac{\eta_1 u_w}{h^2}\right)}{\eta_1 \frac{\partial^2 u_1}{\partial y^2}}, \qquad [11]$$

$$
\begin{array}{ccccccc}
\rho_1 \dfrac{\partial v_1}{\partial t} & + \; \rho_1 u_1 \dfrac{\partial v_1}{\partial x} & + \; \rho_1 v_1 \dfrac{\partial v_1}{\partial y} & = -\dfrac{\partial \phi_1}{\partial y} & + \; \eta_1 \dfrac{\partial^2 v_1}{\partial x^2} & + \; \eta_1 \dfrac{\partial^2 v_1}{\partial y^2} \\
\vdots & \vdots & \vdots & & \vdots & \vdots \\
O\left(\dfrac{\rho_1 u_w^2}{\lambda} \cdot \dfrac{h}{\lambda}\right) & O\left(\dfrac{\rho_1 u_w^2}{\lambda} \cdot \dfrac{h}{\lambda}\right) & O\left(\dfrac{\rho_1 u_w^2}{\lambda} \cdot \dfrac{h}{\lambda}\right) & & O\left(\dfrac{\eta_1 u_w}{h^2} \cdot \dfrac{h^3}{\lambda^3}\right) & O\left(\dfrac{\eta_1 u_w}{h^2} \cdot \dfrac{h}{\lambda}\right)
\end{array}
\qquad [12]
$$

This study is restricted to waves with a wavelength large compared to the distance between the plates, so

$$\frac{h}{\lambda} \ll 1. \qquad [13]$$

Following Yih, in first approximation, all terms that are a factor h/λ smaller than other terms in the differential equations are ignored. This means that the inertial and viscous terms of [12] are ignored, as these terms are a factor h/λ smaller than the corresponding terms in [11]. As the term

$$\eta_1 \frac{\partial^2 u_1}{\partial x^2}$$

is a factor of $(h/\lambda)^2$ smaller than the term

$$\eta_1 \frac{\partial^2 u_1}{\partial y^2},$$

it can of course also be omitted. The ratio of the inertial terms in [11] and the viscous term

$$\eta_1 \frac{\partial^2 u_1}{\partial y^2}$$

of that equation is

$$O\left(\frac{\rho_1 u_w h}{\eta_1} \cdot \frac{h}{\lambda}\right).$$

It is supposed that the value of the Reynolds number $\rho_1 u_w h/\eta_1$ is such that

$$\frac{\rho_1 u_w h}{\eta_1} \cdot \frac{h}{\lambda} \ll 1. \qquad [14]$$

This means that, as in Yih's first approximation, also the inertial terms of [11] can be ignored. In this way [11] and [12] simplify to

$$0 = -\frac{\partial \phi_1}{\partial x} + \eta_1 \frac{\partial^2 u_1}{\partial y^2}, \qquad [15]$$

$$0 = -\frac{\partial \phi_1}{\partial y}. \qquad [16]$$

Equation [16] shows that ϕ_1 is a function of x and t only. Integration of [15] then gives

$$u_1 = \frac{1}{2\eta_1} \frac{\partial \phi_1}{\partial x} y^2 + c_1 y + c_2, \qquad [17]$$

in which c_1 and c_2 are integration constants. With the aid of the boundary conditions

$$\text{for } y = 0 : u_1 = 0 \tag{18}$$

$$\text{for } y = h_1 : u_1 = u_i, \tag{19}$$

it is found that

$$c_1 = \frac{u_i}{h} - \frac{h_1}{2\eta_1}\frac{\partial\phi_1}{\partial x}, \tag{20}$$

$$c_2 = 0. \tag{21}$$

Substitution of [20] and [21] in [17] gives

$$u_1 = \frac{1}{2\eta_1}\frac{\partial\phi_1}{\partial x}y(y - h_1) + u_i\frac{y}{h_1}. \tag{22}$$

From the continuity equation [1] follows

$$\frac{\partial v_1}{\partial y} = -\frac{\partial u_1}{\partial x} \tag{23}$$

or

$$\int_0^{h_1}\frac{\partial v_1}{\partial y}\,\mathrm{d}y = [v_1]_0^{h_1} = v_i = -\int_0^{h_1}\frac{\partial u_1}{\partial x}\,\mathrm{d}y. \tag{24}$$

Substitution of [22] in [24] gives

$$v_i = \frac{h_1^3}{12\eta_1}\frac{\partial^2\phi_1}{\partial x^2} + \frac{h_1^2}{4\eta_1}\frac{\partial h_1}{\partial x}\frac{\partial\phi_1}{\partial x} + \frac{u_i}{2}\frac{\partial h_1}{\partial x} - \frac{h_1}{2}\frac{\partial u_i}{\partial x} \tag{25}$$

or

$$\frac{\partial}{\partial x}\left(h_1^3\frac{\partial\phi_1}{\partial x}\right) = -6\eta_1 u_i\frac{\partial h_1}{\partial x} + 6\eta_1 h_1\frac{\partial u_i}{\partial x} + 12\eta_1 v_i. \tag{26}$$

If also

$$\frac{\rho_2 u_w h}{\eta_2}\cdot\frac{h}{\lambda} \ll 1,$$

it can be shown in the same way that

$$\frac{\partial}{\partial x}\left(h_2^3\frac{\partial\phi_2}{\partial x}\right) = 6\eta_2(u_w - u_i)\frac{\partial h_2}{\partial x} + 6\eta_2 h_2\frac{\partial u_i}{\partial x} - 12\eta_2 v_i. \tag{27}$$

For u_2 an expression similar to [22] for u_1 can be derived, namely,

$$u_2 = \frac{1}{2\eta_2}\frac{\partial\phi_2}{\partial x}y'(y' - h_2) + u_i\frac{y'}{h_2} + u_w\left(1 - \frac{y'}{h_2}\right), \tag{28}$$

in which

$$y' = h - y. \qquad [29]$$

Kinematic and dynamic boundary conditions

At the interface between the two fluids, the kinematic boundary condition holds. This condition can be written as

$$\frac{\mathrm{D}F}{\mathrm{D}t} = 0, \qquad [30]$$

in which D/Dt represents the material derivative and F the equation of the interface

$$F = y - h_1(x, t) = 0. \qquad [31]$$

Substitution of [31] in [30] gives

$$v_i = u_i \frac{\partial h_1}{\partial x} + \frac{\partial h_1}{\partial t}. \qquad [32]$$

The order of magnitude of the terms of [32] are given by

$$\begin{array}{ccccc} v_i & = & u_i \dfrac{\partial h_1}{\partial x} & + & \dfrac{\partial h_1}{\partial t} \\ \vdots & & \vdots & & \vdots \\ O(v) & & O\left(\dfrac{u_w h}{\lambda}\right) & & O\left(\dfrac{u_w h}{\lambda}\right). \end{array} \qquad [33]$$

From [10] it can be concluded that the terms are of the same order of magnitude. So the kinematic boundary condition cannot be simplified. It gives a relation, [32], between the shape h_1 of the interface and the velocity components u_i and v_i of the fluids at the interface.

At the interface of the fluids also the dynamic boundary condition holds. This condition can be written as

$$n_k \sigma_{2,ik} - n_k \sigma_{1,ik} = \gamma \frac{\partial^2 h_1}{\partial x^2} n_i, \text{ for } y = h_1, \qquad [34]$$

in which n_i are the components of the unit normal to the interface, γ the surface tension coefficient and $\sigma_{1,ik}$ and $\sigma_{2,ik}$ the stress tensors for fluids 1 and 2, respectively. $\sigma_{1,ik}$ is given by

$$\sigma_{1,ik} = \left(\begin{array}{c|c|c} -p_1 + 2\eta_1 \dfrac{\partial u_1}{\partial x} & \eta_1\left(\dfrac{\partial u_1}{\partial y} + \dfrac{\partial v_1}{\partial x}\right) & 0 \\ \hline \eta_1\left(\dfrac{\partial v_1}{\partial x} + \dfrac{\partial u_1}{\partial y}\right) & -p_1 + 2\eta_1 \dfrac{\partial v_1}{\partial y} & 0 \\ \hline 0 & 0 & -p_1 \end{array} \right). \qquad [35]$$

The order of magnitude of the terms in [35] will be determined. From [7], [11] and [15] follows

$$p_1 = O\left(\frac{\eta_1 u_w \lambda}{h^2}\right). \qquad [36]$$

The other terms can be estimated with the aid of [10]; it is then found that

$$2\eta_1 \frac{\partial u_1}{\partial x} = O\left(\frac{\eta_1 u_w \lambda}{h^2} \cdot \frac{h^2}{\lambda^2}\right), \qquad [37]$$

$$\eta_1 \frac{\partial u_1}{\partial y} = O\left(\frac{\eta_1 u_w \lambda}{h^2} \cdot \frac{h}{\lambda}\right), \qquad [38]$$

$$\eta_1 \frac{\partial v_1}{\partial x} = O\left(\frac{\eta_1 u_w \lambda}{h^2} \cdot \frac{h^3}{\lambda^3}\right), \qquad [39]$$

$$2\eta_1 \frac{\partial v_1}{\partial y} = O\left(\frac{\eta_1 u_2 \lambda}{h^2} \cdot \frac{h^2}{\lambda^2}\right). \qquad [40]$$

For $\sigma_{2,ik}$, a similar estimate can be made.

The ratio of the two components n_1 and n_2 of the unit normal can be estimated as

$$\frac{n_2}{n_1} = O\left(\frac{h}{\lambda}\right). \qquad [41]$$

Substitution of [35]–[41] in [34] gives in first approximation

$$-(p_1 - p_2)n_1 + \left(\eta_1 \frac{\partial u_1}{\partial y} - \eta_2 \frac{\partial u_2}{\partial y}\right) n_2 = \gamma \frac{\partial^2 h_1}{\partial x^2} n_1, \qquad \text{for } y = h_1, \qquad [42]$$

$$\left(\eta_1 \frac{\partial u_1}{\partial y} - \eta_2 \frac{\partial u_2}{\partial y}\right) n_1 - (p_1 - p_2)n_2 = \gamma \frac{\partial^2 h_1}{\partial x^2} n_2, \qquad \text{for } y = h_1. \qquad [43]$$

These relations must be satisfied for any combination (n_1, n_2). That is possible only if

$$-(p_1 - p_2) = \gamma \frac{\partial^2 h_1}{\partial x^2}, \qquad \text{for } y = h_1, \qquad [44]$$

$$\eta_1 \frac{\partial u_1}{\partial y} - \eta_2 \frac{\partial u_2}{\partial y} = 0, \qquad \text{for } y = h_1. \qquad [45]$$

The ratio of the surface tension term and pressure terms in [44] is respectively of the following order of magnitude:

$$O\left(\frac{\gamma}{\eta_1 u_w} \cdot \frac{h^3}{\lambda^3}\right) \quad \text{and} \quad O\left(\frac{\gamma}{\eta_2 u_w} \cdot \frac{h^3}{\lambda^3}\right).$$

If it is assumed that

$$\frac{\gamma}{\eta_1 u_w} \cdot \frac{h^3}{\lambda^3} \ll 1, \qquad [46]$$

$$\frac{\gamma}{\eta_2 u_w} \cdot \frac{h^3}{\lambda^3} \ll 1, \quad [47]$$

equation [44] reduces to

$$(p_1)_{y=h_1} = (p_2)_{y=h_1}. \quad [48]$$

Substitution of [7] and [8] gives

$$\phi_1 = \phi_2 + (\rho_1 - \rho_2) g h_1. \quad [49]$$

Substitution of [22] and [28] in [45] gives

$$\frac{h_1}{2}\frac{\partial \phi_1}{\partial x} + \frac{\eta_1 u_i}{h_1} = -\frac{h_2}{2}\frac{\partial \phi_2}{\partial x} - \frac{\eta_2 (u_i - u_w)}{h_2}. \quad [50]$$

This can be written as

$$u_i = \frac{1}{(\eta_1/h_1 + \eta_2/h_2)} \cdot \left(-\frac{h_1}{2}\frac{\partial \phi_1}{\partial x} - \frac{h_2}{2}\frac{\partial \phi_2}{\partial x} + u_w \frac{\eta_2}{h_2} \right). \quad [51]$$

Equations [49] and [51] are the dynamic boundary conditions.

Complete set of equations in dimensionless form

The flow problem is described by [26], [27], [32], [49] and [51], which for the sake of completeness are repeated here:

$$\frac{\partial}{\partial x}\left(h_1^3 \frac{\partial \phi_1}{\partial x} \right) = -6\eta_1 u_i \frac{\partial h_1}{\partial x} + 6\eta_1 h_1 \frac{\partial u_i}{\partial x} + 12\eta_1 v_i, \quad [26]$$

$$\frac{\partial}{\partial x}\left(h_2^3 \frac{\partial \phi_2}{\partial x} \right) = 6\eta_2 (u_w - u_i) \frac{\partial h_2}{\partial x} + 6\eta_2 h_2 \frac{\partial u_i}{\partial x} - 12\eta_2 v_i, \quad [27]$$

$$v_i = u_i \frac{\partial h_1}{\partial x} + \frac{\partial h_1}{\partial t}, \quad [32]$$

$$\phi_1 = \phi_2 + (\rho_1 - \rho_2) g h_1, \quad [49]$$

$$u_i = \frac{1}{(\eta_1/h_1 + \eta_2/h_2} \cdot \left(-\frac{h_1}{2}\frac{\partial \phi_1}{\partial x} - \frac{h_2}{2}\frac{\partial \phi_2}{\partial x} + u_w \frac{\eta_2}{h_2} \right). \quad [51]$$

This set of equations is complemented by the relation

$$h_1 + h_2 = h. \quad [52]$$

The set of equations is written in dimensionless form by introducing the following dimensionless quantities,

$$H_1 = \frac{h_1}{h}, \quad H_2 = \frac{h_2}{h}, \quad X = \frac{x}{h}, \quad Y = \frac{y}{h}, \quad U_i = \frac{u_i}{u_w},$$
$$V_i = \frac{v_i}{u_w}, \quad T = \frac{t u_w}{h}, \quad \Phi_1 = \frac{\phi_1 h}{\eta_1 u_w}, \quad \Phi_2 = \frac{\phi_2 h}{\eta_1 u_w}, \quad [53]$$

and the following dimensionless groups,

$$C_1 = \frac{\eta_2}{\eta_1}, \quad [54]$$

$$C_2 = \frac{(\rho_1 - \rho_2) g h^2}{\eta_1 u_w} \quad [55]$$

Substitution in the set of equations gives

$$\frac{\partial}{\partial X}\left(H_1^3 \frac{\partial \Phi_1}{\partial X}\right) = -6U_i \frac{\partial H_1}{\partial X} + 6H_1 \frac{\partial U_i}{\partial X} + 12V_i, \quad [56]$$

$$\frac{\partial}{\partial X}\left(H_2^3 \frac{\partial \Phi_2}{\partial X}\right) = 6C_1(1 - U_i)\frac{\partial H_2}{\partial X} + 6C_1 H_2 \frac{\partial U_i}{\partial X} - 12C_1 V_i, \quad [57]$$

$$V_i = U_i \frac{\partial H_1}{\partial X} + \frac{\partial H_1}{\partial T}, \quad [58]$$

$$\Phi_1 = \Phi_2 + C_2 H_1, \quad [59]$$

$$U_i = \frac{1}{(1/H_1 + C_1/H_2)} \cdot \left(-\frac{H_1}{2}\frac{\partial \Phi_1}{\partial X} - \frac{H_2}{2}\frac{\partial \phi_2}{\partial X} + \frac{C_1}{H_2}\right), \quad [60]$$

$$H_1 + H_2 = 1. \quad [61]$$

So far we have not considered the imposed pressure gradient. It is now assumed that the dimensionless pressure drop over one wavelength is equal to C_3. So C_3 is defined as

$$C_3 = \Phi_1(X, T) - \Phi_1(X + L, T), \quad [62]$$

in which L is the dimensionless wavelength

$$L = \frac{\lambda}{h}. \quad [63]$$

With the definition of the dimensionless quantities given in [53], C_3 can be written as

$$C_3 = \frac{h}{\eta_1 u_w}[\phi_1(x, t) - \phi_1(x + \lambda, t)]. \quad [64]$$

Single equation for the shape of the wave at the interface

The purpose of the coming calculation is to reduce the set of equations [56]–[61] to one equation. First, [56] is multiplied by C_1 and added to [57]. This yields after some computations

$$\frac{\partial}{\partial X}\left(C_1 H_1^3 \frac{\partial \Phi_1}{\partial X} + H_2^3 \frac{\partial \Phi_2}{\partial X}\right) = \frac{\partial}{\partial X}(6C_1 H_2 + 6C_1 U_i). \quad [65]$$

Integration of [65] gives

$$C_1 H_1^3 \frac{\partial \Phi_1}{\partial X} + H_2^3 \frac{\partial \Phi_2}{\partial X} = 6C_1 H_2 + 6C_1 U_i + G, \quad [66]$$

in which G is a function of Y and T only. Substitution of [59] in [66] yields

$$(C_1H_1^3 + H_2^3)\frac{\partial \Phi_1}{\partial X} - 6C_1U_i = C_2H_2^3\frac{\partial H_1}{\partial X} + 6C_1H_2 + G. \qquad [67]$$

Next, [59] is substituted in [60]. It is then found that

$$\frac{H_1H_2}{2(H_2 + C_1H_1)} \cdot \frac{\partial \Phi_1}{\partial X} + U_i = \frac{C_2H_1H_2^2}{2(H_2 + C_1H_1)}\frac{\partial H_1}{\partial X} + \frac{C_1H_1}{H_2 + C_1H_1}. \qquad [68]$$

Substitution of [58] in [56] gives after some computations

$$\frac{\partial H_1}{\partial T} = \frac{\partial}{\partial X}\left(\frac{H_1^3}{12}\frac{\partial \Phi_1}{\partial X} - \frac{H_1U_i}{2}\right). \qquad [69]$$

The flow problem is now reduced to solving [67]–[69] with the additional relation [61]. For the ease of writing, the following quantities are introduced:

$$\alpha_1 = C_1H_1^3 + H_2^3, \qquad \alpha_2 = \frac{H_1H_2}{2(H_2 + C_1H_1)}, \qquad [70]$$

$$\beta_1 = -6C_1, \qquad \beta_2 = 1, \qquad [71]$$

$$\gamma_1 = C_2H_2^3, \qquad \gamma_2 = \frac{C_2H_1H_2^2}{2(H_2 + C_1H_1)}, \qquad [72]$$

$$\delta_1 = 6C_1H_2, \qquad \delta_2 = \frac{C_1H_1}{H_2 + C_1H_1}. \qquad [73]$$

Equations [67] and [68] can now be written as

$$\alpha_1\frac{\partial \Phi_1}{\partial X} + \beta_1U_i = \gamma_1\frac{\partial H_1}{\partial X} + \delta_1 + G \qquad [74]$$

$$\alpha_2\frac{\partial \Phi_1}{\partial X} + \beta_2U_i = \gamma_2\frac{\partial H_1}{\partial X} + \delta_2. \qquad [75]$$

The solution of [74] and [75] is

$$\frac{\partial \Phi_1}{\partial X} = \frac{\beta_2\gamma_1 - \beta_1\gamma_2}{\alpha_1\beta_2 - \alpha_2\beta_1} \cdot \frac{\partial H_1}{\partial X} + \frac{\beta_2\delta_1 - \beta_1\delta_2}{\alpha_1\beta_2 - \alpha_2\beta_1} + \frac{\beta_2}{\alpha_1\beta_2 - \alpha_2\beta_1}G, \qquad [76]$$

$$U_i = \frac{\alpha_1\gamma_2 - \alpha_2\gamma_1}{\alpha_1\beta_2 - \alpha_2\beta_1} \cdot \frac{\partial H_1}{\partial X} + \frac{\alpha_1\delta_2 - \alpha_2\delta_1}{\alpha_1\beta_2 - \alpha_2\beta_1} - \frac{\alpha_2}{\alpha_1\beta_2 - \alpha_2\beta_1}G. \qquad [77]$$

Again, for ease of writing, the following quantities are introduced:

$$\lambda_1 = \frac{\beta_2\gamma_1 - \beta_1\gamma_2}{\alpha_1\beta_2 - \alpha_2\beta_1}, \qquad \lambda_2 = \frac{\alpha_1\gamma_2 - \alpha_2\gamma_1}{\alpha_1\beta_2 - \alpha_2\beta_1}, \qquad [78]$$

$$\mu_1 = \frac{\beta_2\delta_1 - \beta_1\delta_2}{\alpha_1\beta_2 - \alpha_2\beta_1}, \qquad \mu_2 = \frac{\alpha_1\delta_2 - \alpha_2\delta_1}{\alpha_1\beta_2 - \alpha_2\beta_1}, \qquad [79]$$

$$\omega_1 = \frac{\beta_2}{\alpha_1\beta_2 - \alpha_2\beta_1}, \qquad \omega_2 = \frac{-\alpha_2}{\alpha_1\beta_2 - \alpha_2\beta_1}, \tag{80}$$

equations [76] and [77] can now be written as

$$\frac{\partial \Phi_1}{\partial X} = \lambda_1 \frac{\partial H_1}{\partial X} + \mu_1 + \omega_1 G, \tag{81}$$

$$U_i = \lambda_2 \frac{\partial H_1}{\partial X} + \mu_2 + \omega_2 G. \tag{82}$$

Substitution of [81] and [82] in [69] yields after some computations

$$\frac{\partial H_1}{\partial T} = \left(\lambda_1 \frac{H_1^3}{12} - \lambda_2 \frac{H_1}{2}\right)\frac{\partial^2 H_1}{\partial X^2} + \frac{\partial H_1}{\partial X} \cdot \frac{\partial}{\partial X}\left(\lambda_1 \frac{H_1^3}{12} - \lambda_2 \frac{H_1}{2}\right) + \frac{\partial}{\partial X}\left(\mu_1 \frac{H_1^3}{12} - \mu_2 \frac{H_1}{2}\right) + G \cdot \frac{\partial}{\partial X}\left(\omega_1 \frac{H_1^3}{12} - \omega_2 \frac{H_1}{2}\right). \tag{83}$$

Finally, G has to be determined. First of all, G is independent of Y, as all terms of [67] are independent of Y. So G is a function of T only. Integration of [81] over one wavelength gives after some calculations, and keeping in mind that

$$\int_{X_0}^{X_0+L} \frac{\partial \Phi_1}{\partial X} \mathrm{d}X = \Phi_1(X_0 + L, T) - \Phi_1(X_0, T) = -C_3, \tag{84}$$

the following expression for G:

$$G = -\frac{C_3 + \int_{X_0}^{X_0+L}\left(\lambda_1 \frac{\partial H_1}{\partial X} + \mu_1\right)\mathrm{d}X}{\int_{X_0}^{X_0+L} \omega_1 \mathrm{d}X}. \tag{85}$$

After substitution of [85] in [83], we finally find

$$\frac{\partial H_1}{\partial T} = \left(\lambda_1 \frac{H_1^3}{12} - \lambda_2 \frac{H_1}{2}\right)\frac{\partial^2 H_1}{\partial X^2} + \frac{\partial H_1}{\partial X} \cdot \frac{\partial}{\partial X}\left(\lambda_1 \frac{H_1^3}{12} - \lambda_2 \frac{H_1}{2}\right) + \frac{\partial}{\partial X}\left(\mu_1 \frac{H_1^3}{12} - \mu_2 \frac{H_1}{2}\right) \frac{- C_3 + \int_{X_0}^{X_0+L}\left(\lambda_1 \frac{\partial H_1}{\partial X} + \mu_1\right)\mathrm{d}X}{\int_{X_0}^{X_0+L} \omega_1 \,\mathrm{d}X} \cdot \frac{\partial}{\partial X}\left(\omega_1 \frac{H_1^3}{12} - \omega_2 \frac{H_1}{2}\right). \tag{86}$$

Only one unknown quantity (H_1) is present in [86]. $H_1(X, T)$ gives the shape of the wave at the interface. Starting from an initial condition for H_1, the development in time of the shape of this wave can be calculated with the aid of [86] as a function of the parameters C_1, C_2 and C_3. We have simulated this development in time on a computer by using [86] in the form of a finite difference equation. Forward time and centered space differences have been used; when a rather steep slope developed in the shape of the wave, an upwind differencing method was applied. Special attention was paid to the value of the time step ΔT and the length step ΔX. For every simulation, the values of these steps were decreased until the results of the simulation did not change anymore. For the simulation, we started with an initial wave that had the shape of a sine function,

$$H_1^{(0)} = \overline{H}_1 + A^{(0)} \sin \frac{2\pi X}{L}, \qquad [87]$$

in which $\overline{H}_1 = \overline{h}_1/h$ gives the average dimensionless thickness of the layer of fluid 1, $A^{(0)} = a^{(0)}/h$ represents the dimensionless initial amplitude of the wave and $L = \lambda/h$ is the dimensionless wavelength.

3. RESULTS

To check the correctness of the theory and the computer program, it was first assumed that the viscosity of fluid 1 was 10^6 times larger than the viscosity of fluid 2, so $C_1 = 10^{-6}$. For such a low value of the viscosity ratio, it can be expected that the influence of fluid 2 is

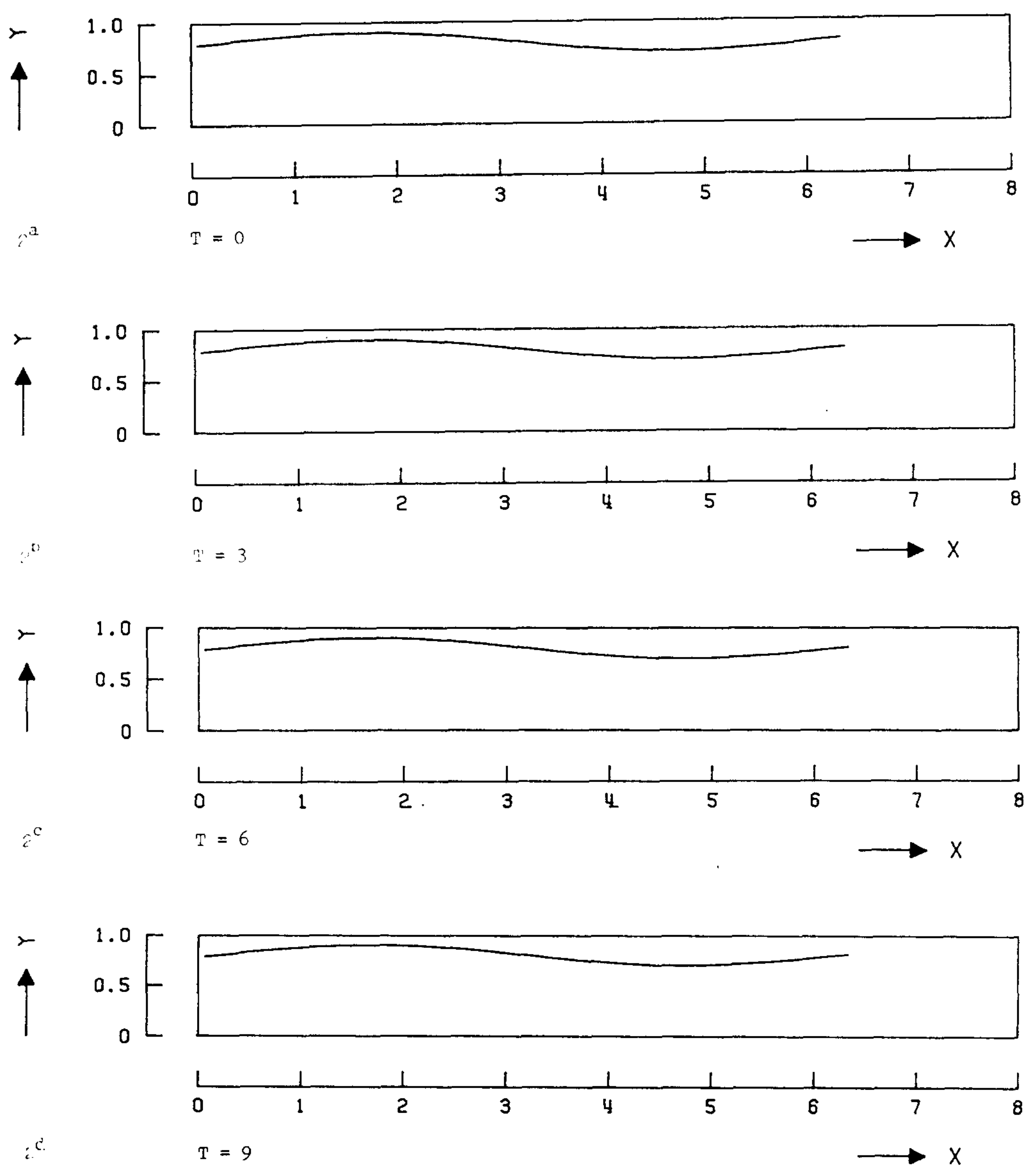

Figure 2. Development of interfacial wave with time for extremely low viscosity ratio. $C_1 = 10^{-6}$; $C_2 = 0$; $C_3 = 0$; $\overline{H}_1 = 0.8$; $A^{(0)} = 0.1$; $L = 6$; $\Delta T = 0.12$; $\Delta X = 0.066$.

negligible, and that the shape of the wave will not change in time. In figure 2, the result of the numerical simulation for this case is presented; the development of the interfacial wave is calculated from $T = 0$ up to $T = 21$. As can be seen, the simulation is in agreement with the expectation. In the figure, the values of the parameters C_1, C_2, C_3, $\overline{H}_1$, $A^{(0)}$, L, ΔT and ΔX are given.

Next, the viscosity ratio C_1 was increased. Already at a value of 10^{-5}, the wave started to develop and move as function of time. In figure 3, the result of the simulation for $C_1 = 10^{-4}$ is given. As can be seen, a sawtoothlike shape of the wave starts to develop, which does not change anymore after a certain time. This equilibrium wave continues to travel with a constant velocity in the x-direction. In figure 4, the results for $C_1 = 10^{-1}$ are given; the same conclusion as for the case of $C_1 = 10^{-4}$ can be drawn.

So far, the dimensionless initial amplitude of the wave has had the value of $A^{(0)} = 0.1$. To investigate the wave development as a function of the value of the initial amplitude, we kept

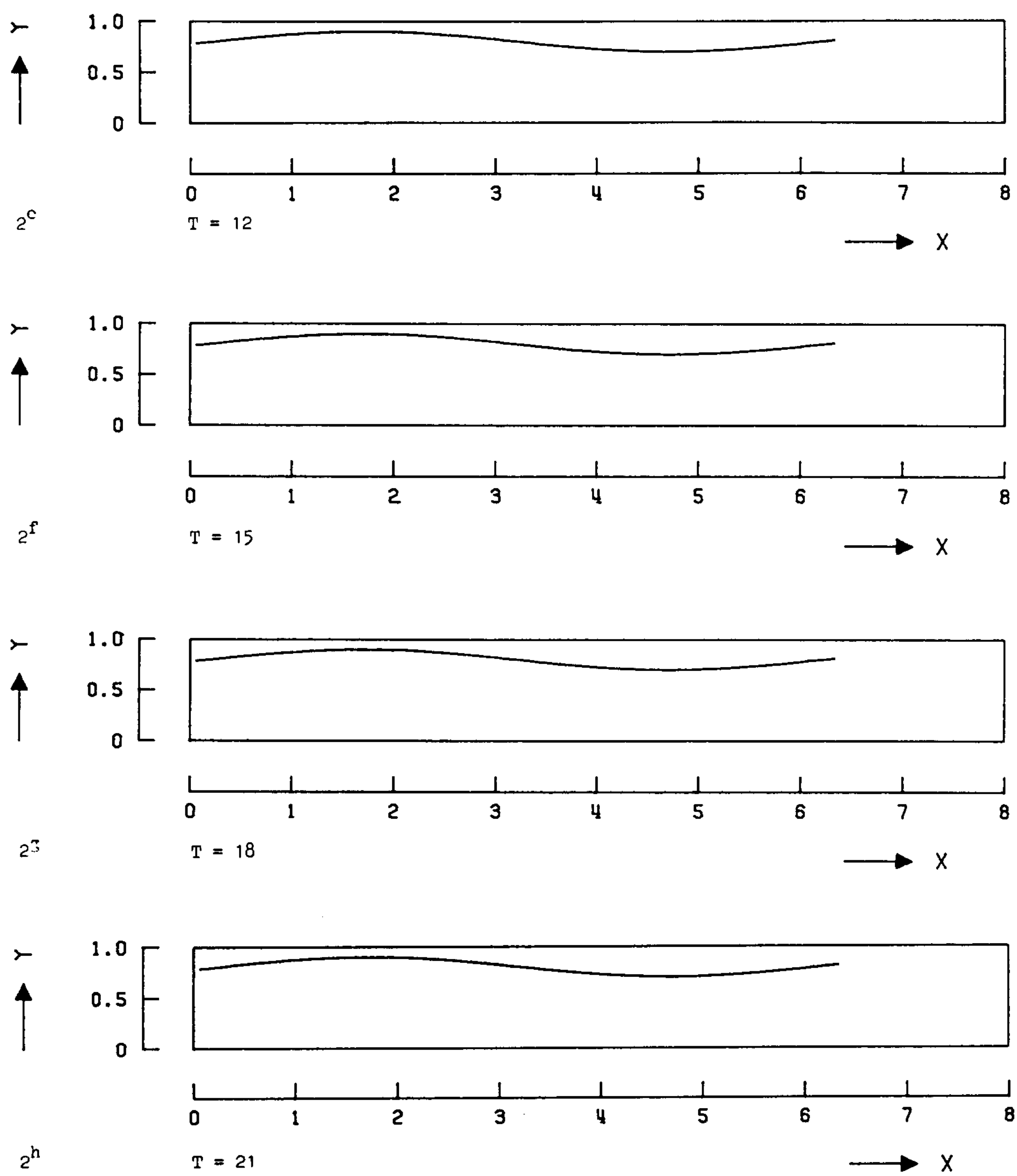

Figure 2. Continued.

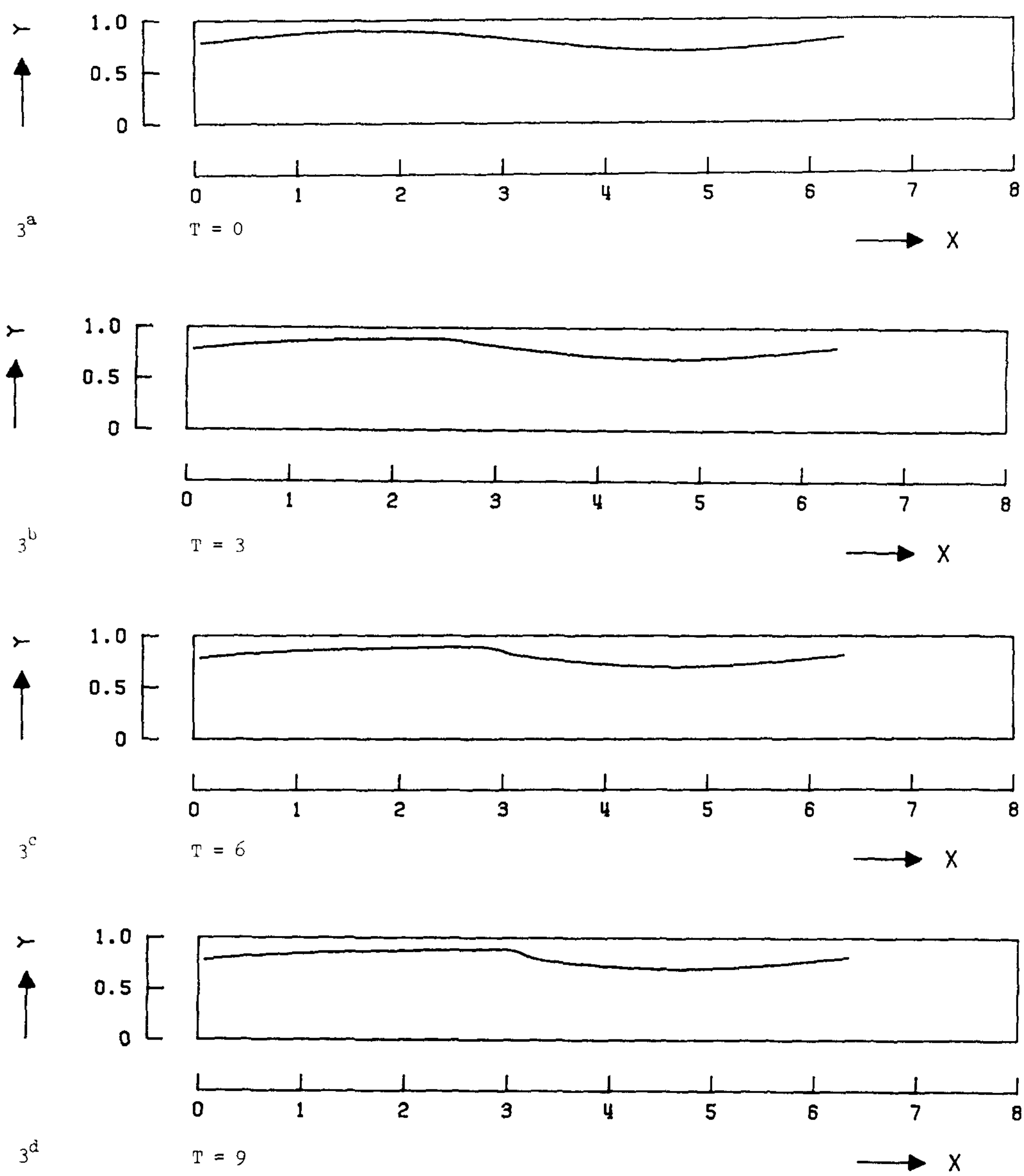

Figure 3. Development of interfacial wave with time for viscosity ratio of 10^{-4}. $C_1 = 10^{-4}$; $C_2 = 0$; $C_3 = 0$; $\overline{H}_1 = 0.8$; $A^{(0)} = 0.1$; $L = 6$; $\Delta T = 0.12$; $\Delta X = 0.066$.

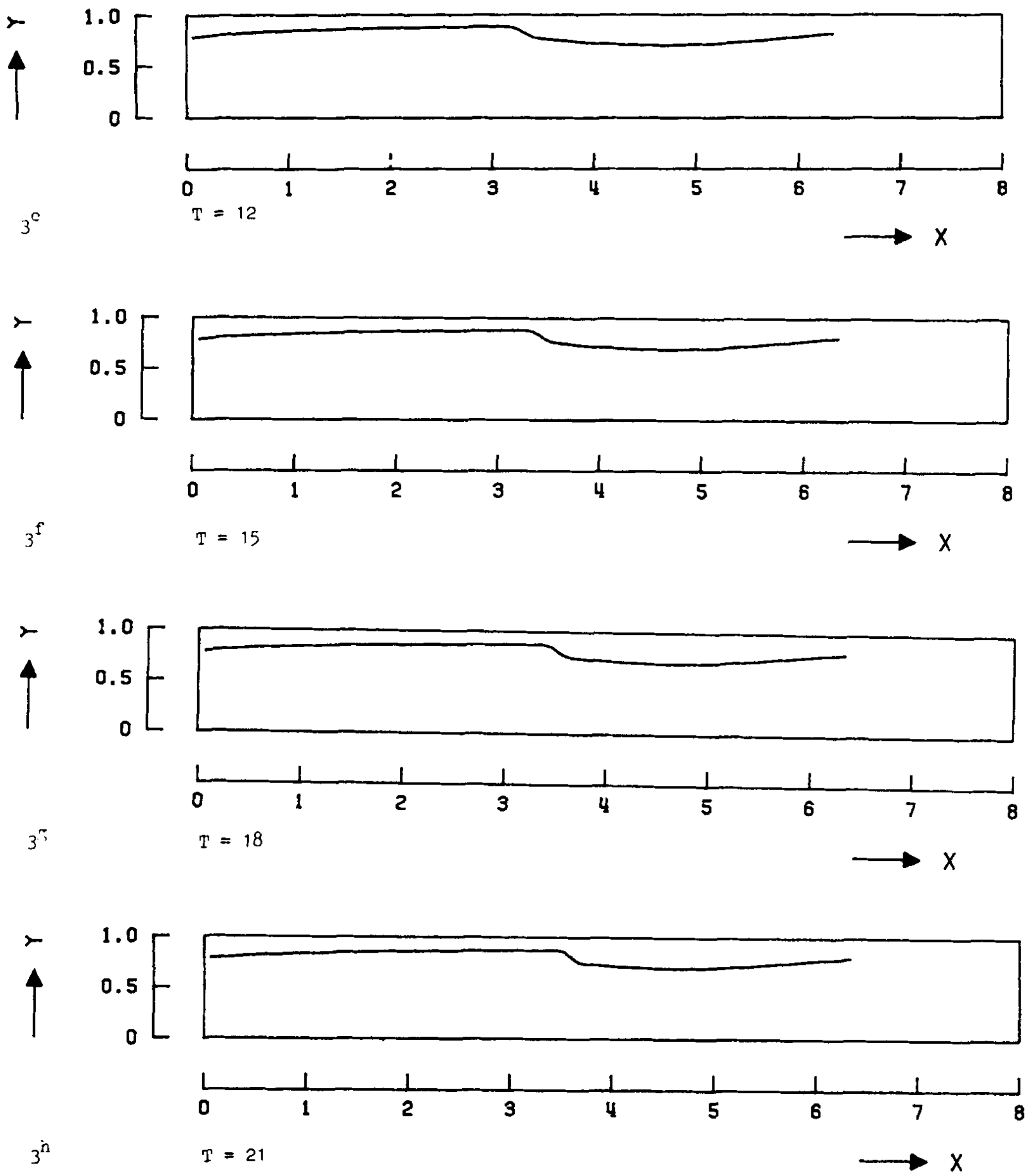

Figure 3. Continued.

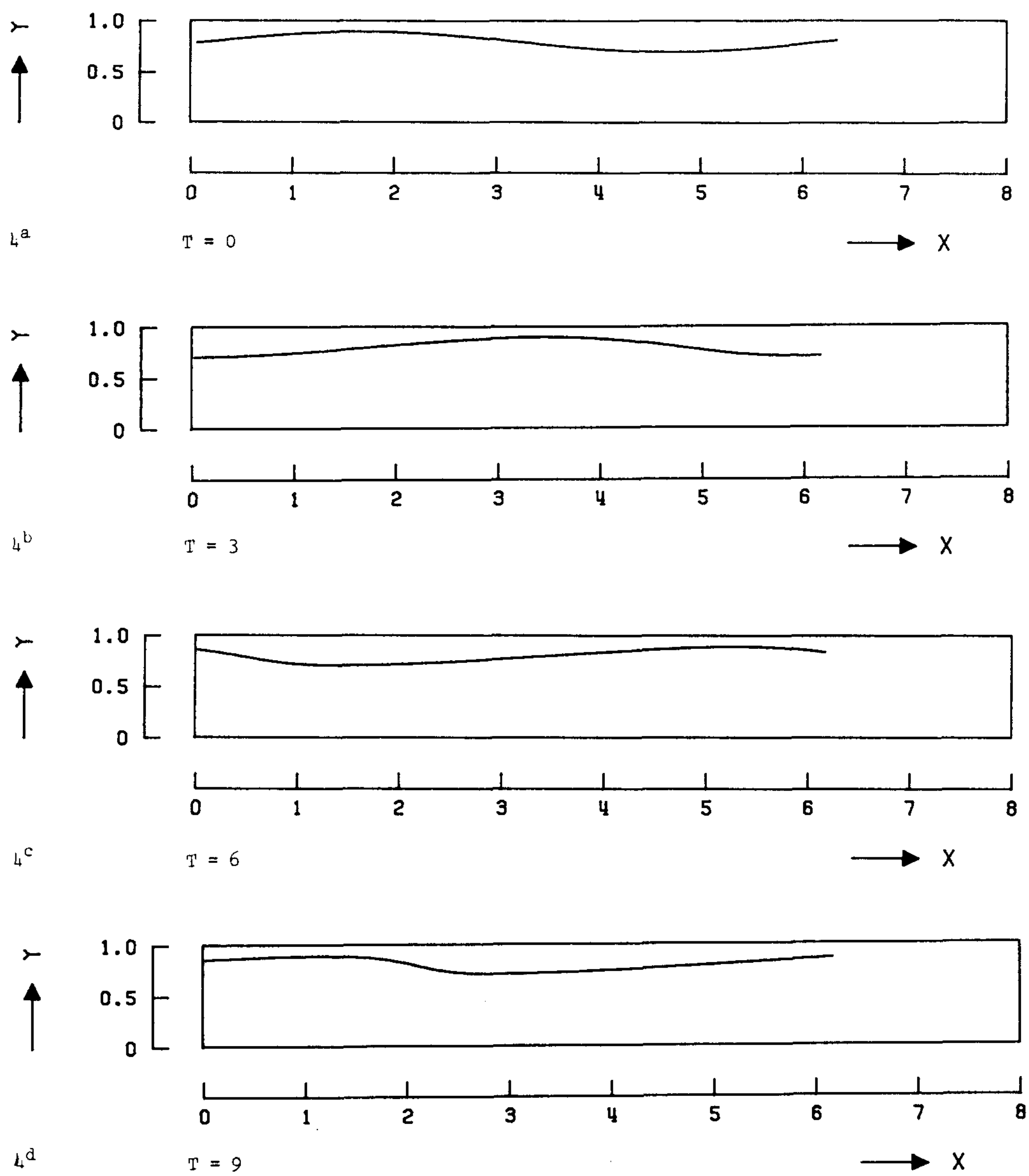

Figure 4. Development of interfacial wave with time for viscosity ratio of 10^{-1}. $C_1 = 0.1$; $C_2 = 0$; $C_3 = 0$; $\overline{H}_1 = 0.8$; $A^{(0)} = 0.1$; $L = 6$; $\Delta T = 0.06$; $\Delta X = 0.033$.

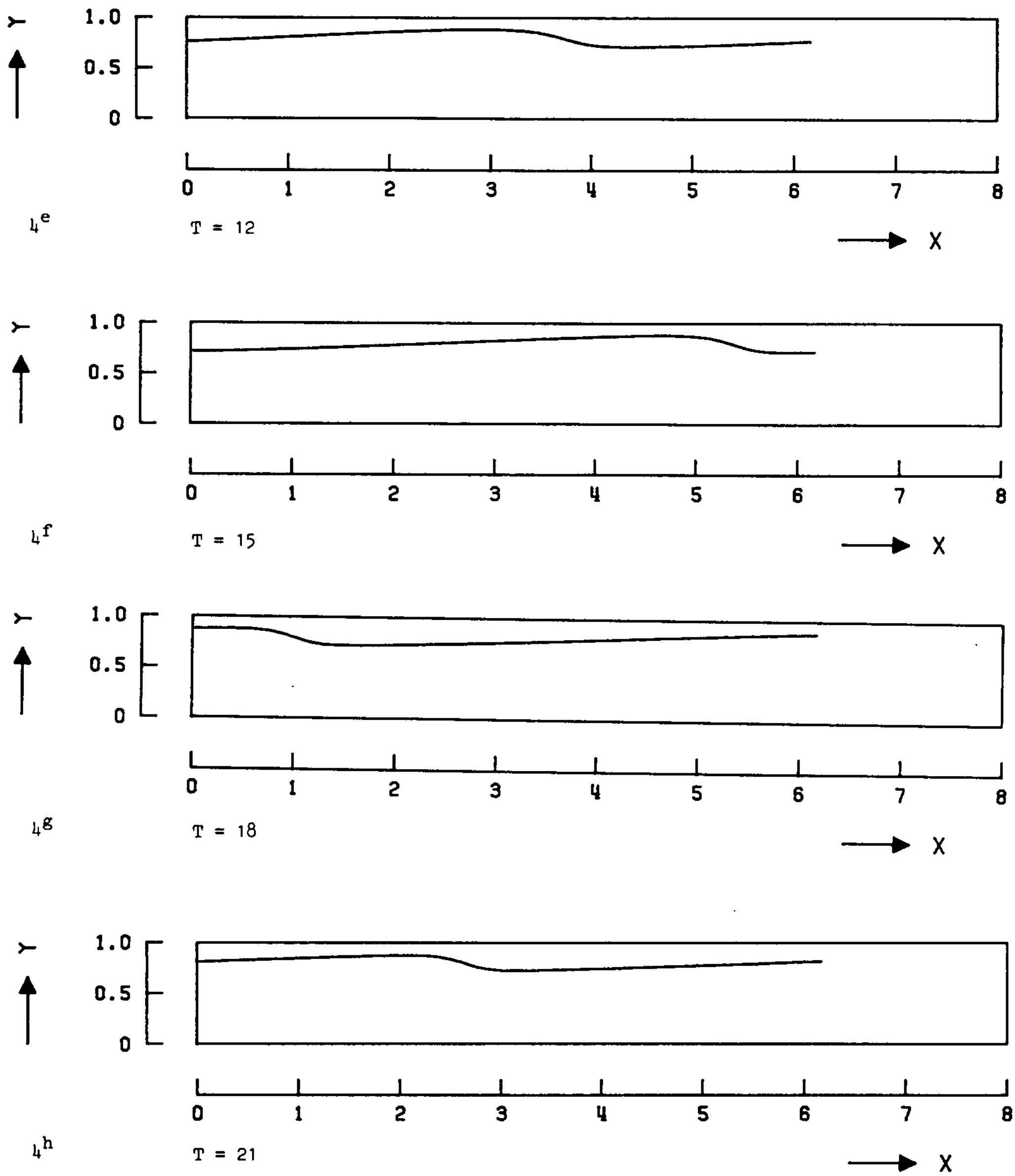

Figure 4. Continued.

the viscosity ratio at the value of $C_1 = 10^{-4}$ and varied $A^{(0)}$ from 0.02 up to 0.15. It was found that for very small values of $A^{(0)}$ (for instance, $A^{(0)} = 0.02$ or 0.04), the shape of the wave did not change at all. The wave propagated at constant velocity in the x-direction without disturbance. The velocity was equal to the velocity as found by Yih in first approximation. So for very small wave amplitudes, our simulation is in agreement with Yih's results. This is another indication for the correctness of our theory and computer program. For larger values of $A^{(0)}$ (for instance, $A^{(0)} = 0.07$), the wave started to develop the sawtoothlike shape as already shown in figure 3; however the amplitude of the wave remained nearly constant in time. This can also be seen in figure 5, where the maximum value $H_1^{(max)}$ of H_1 is given as function of T. For still larger values of $A^{(0)}$, it was found that $H_1^{(max)}$ decreased with time to a certain equilibrium value that depends on $A^{(0)}$ (see also figure 5). When the wave top comes too close to the upper plate, it is pushed back by pressure and viscous forces.

Next, the influence of a difference in density between the two fluids on the wave development was studied. For $\rho_1 > \rho_2$, a damping of the wave may be expected. To that purpose, the value of C_2 was gradually increased from 0 to 0.5, keeping all other parameters constant. For very small values of C_2 (for instance, $C_2 = 0.01$), there was no influence of the density difference on the wave development and wave velocity. However, for larger values of C_2 (for instance, $C_2 = 1$), a quick damping of the wave amplitude was observed. An example is given in figure 6 for $C_2 = 0.5$, where the wave has almost completely disappeared at $T = 21$.

Finally, the influence of an imposed pressure gradient was investigated. Keeping all other parameters constant, C_3 was decreased from about 0.1 to about −0.1. For a positive value of C_3 (pressure decreases in x-direction), the movement of the wave was not observed to be essentially different from the case in which $C_3 = 0$. Only the equilibrium wave amplitude and velocity changed. For a negative value of C_3 (pressure increases in x-direction), a strange phenomenon was observed. For instance, for $C_3 = -0.1$, again a sawtoothlike shape developed; however, this time the sawtoothlike shape was in the opposite direction as for the case of $C_3 = 0$ and the wave started to move in the $-x$-direction. In figure

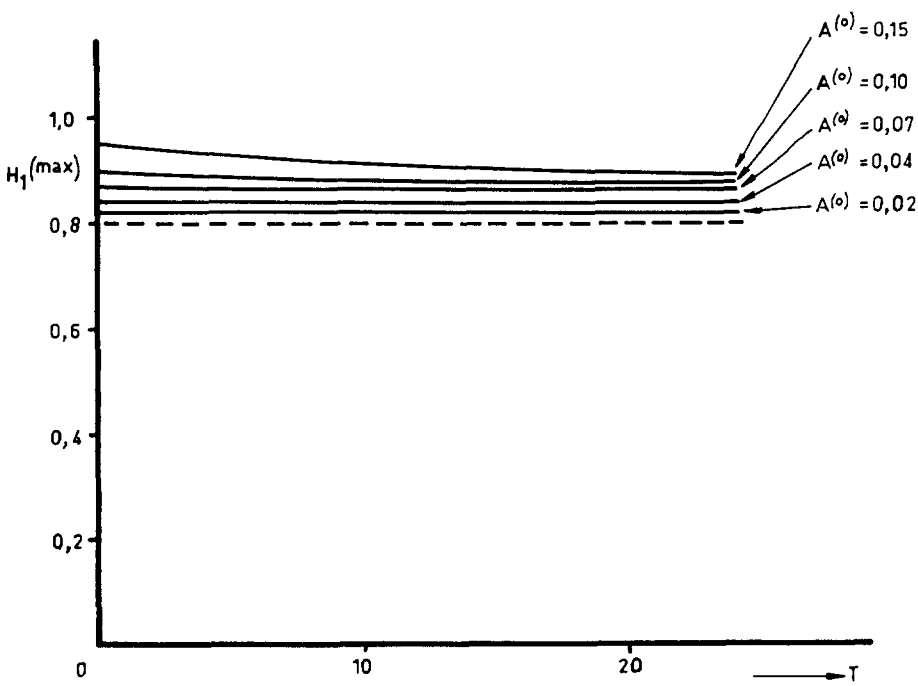

Figure 5. Maximum thickness of layer of fluid 1 as a function of time for various values of the initial amplitude. $C_1 = 10^{-4}$; $C_2 = 0$; $C_3 = 0$; $\overline{H}_1 = 0.8$; $L = 6$; $\Delta T = 0.12$; $\Delta X = 0.066$.

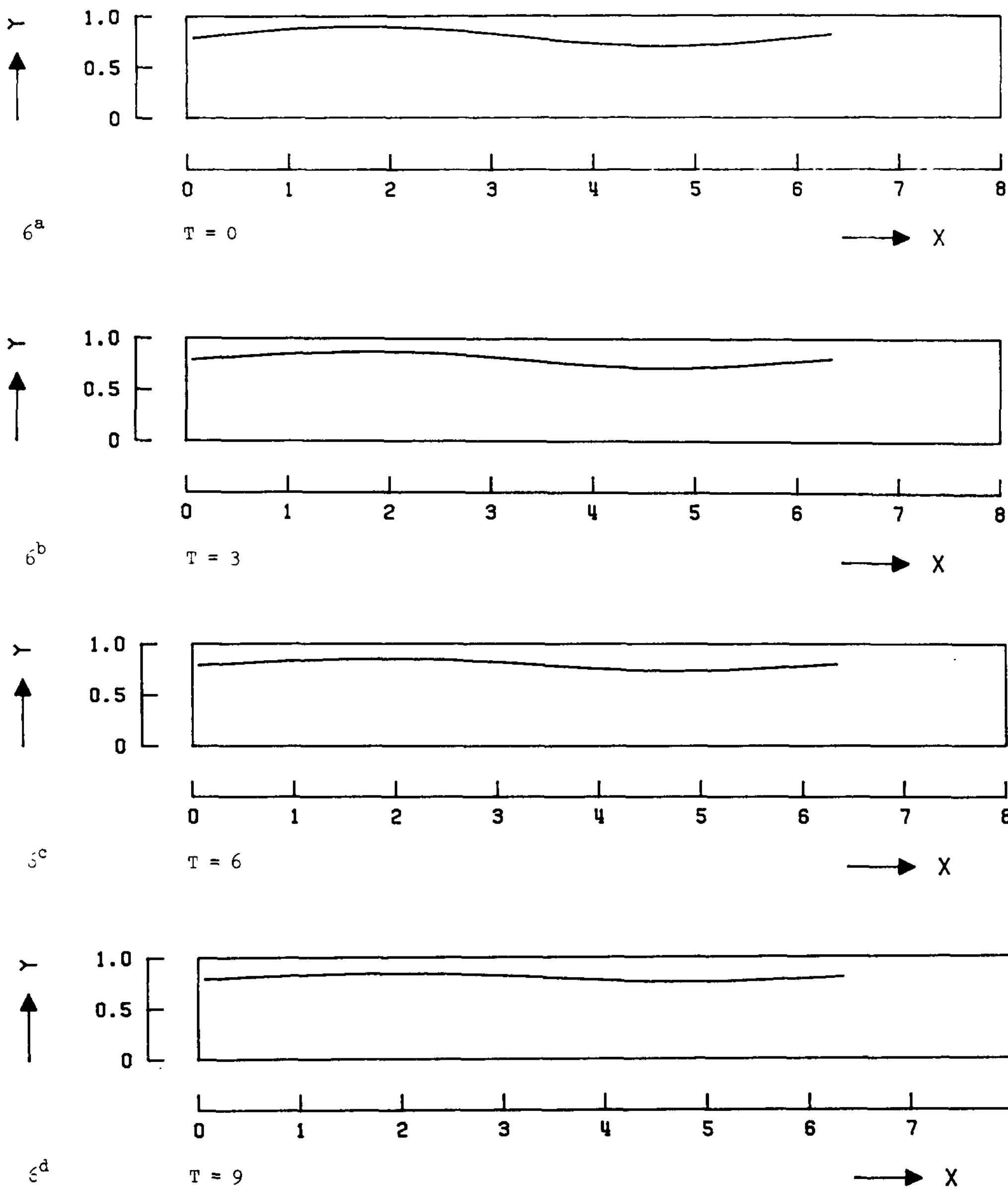

Figure 6. Development of interfacial wave with time for a large density difference. $C_1 = 10^{-4}$; $C_2 = 0.5$; $C_3 = 0$; $\overline{H}_1 = 0.8$; $A^{(0)} = 0.1$; $L = 6$; $\Delta T = 0.02$; $\Delta X = 0.066$.

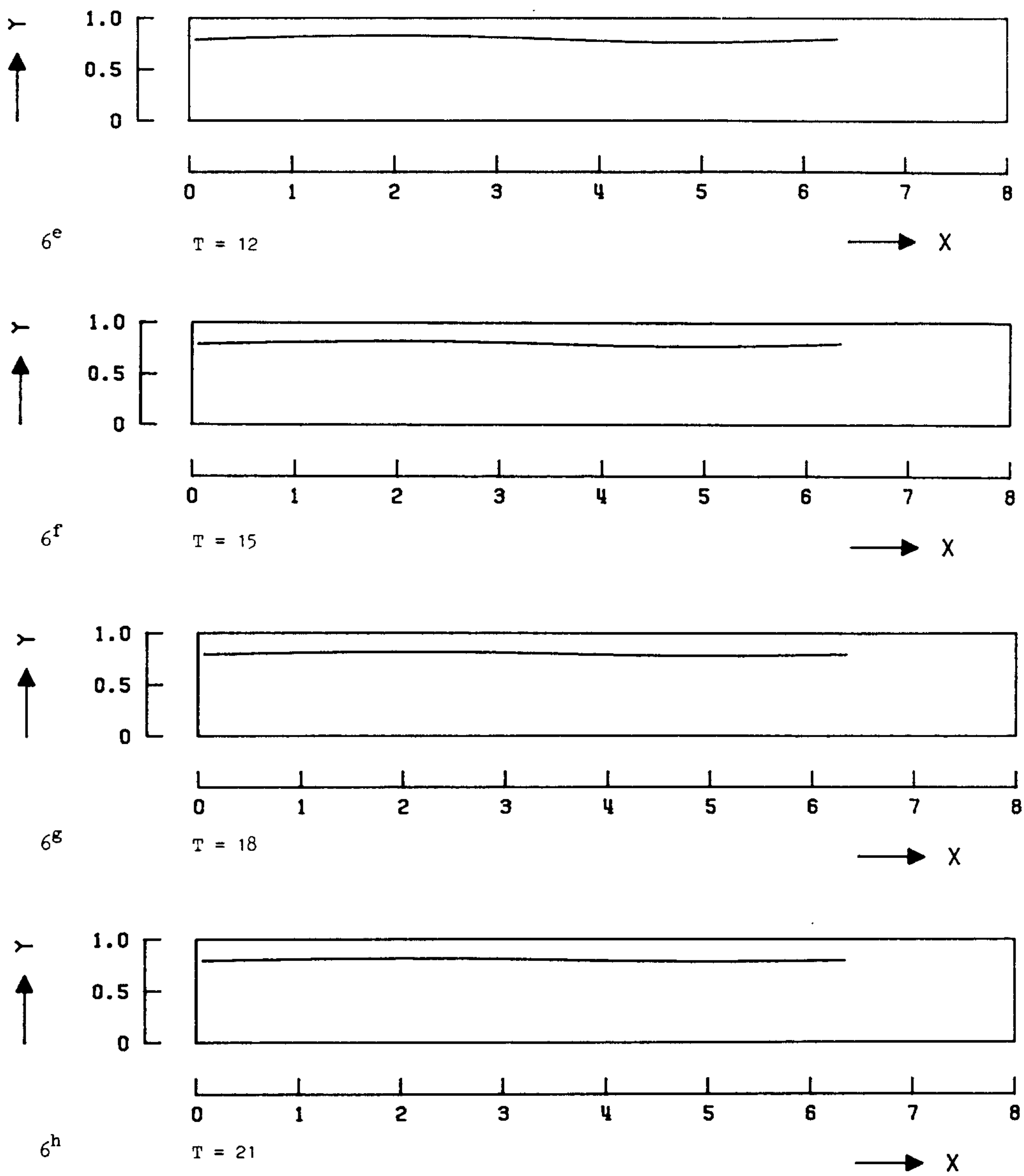

Figure 6. Continued.

7, the simulation for $C_3 = -0.1$ is shown. Obviously, for such values of C_3, the imposed pressure gradient has a stronger influence on the development and movement of the wave than the shear due to the movement of the upper plate.

4. CONCLUSION

We have shown that long waves of finite amplitude are possible at the interface of a plane Couette-Poiseuille flow of two superposed layers of fluids between horizontal plates. For the finite amplitude waves, the existence of which was already suggested by Yih in 1967, an equation has been derived for the development in time of their shape. In deriving this equation from the Navier-Stokes equations, we have assumed that the ratio of plate distance and wavelength and the product of the Reynolds numbers for the two layers and this ratio are small. Moreover, to simplify the dynamic boundary conditions, we have assumed that surface tension terms are small compared to viscous terms.

An imposed symmetric wave disturbance develops in time into a stable asymmetric finite amplitude wave, provided the initial amplitude exceeds a certain value. The equilibrium

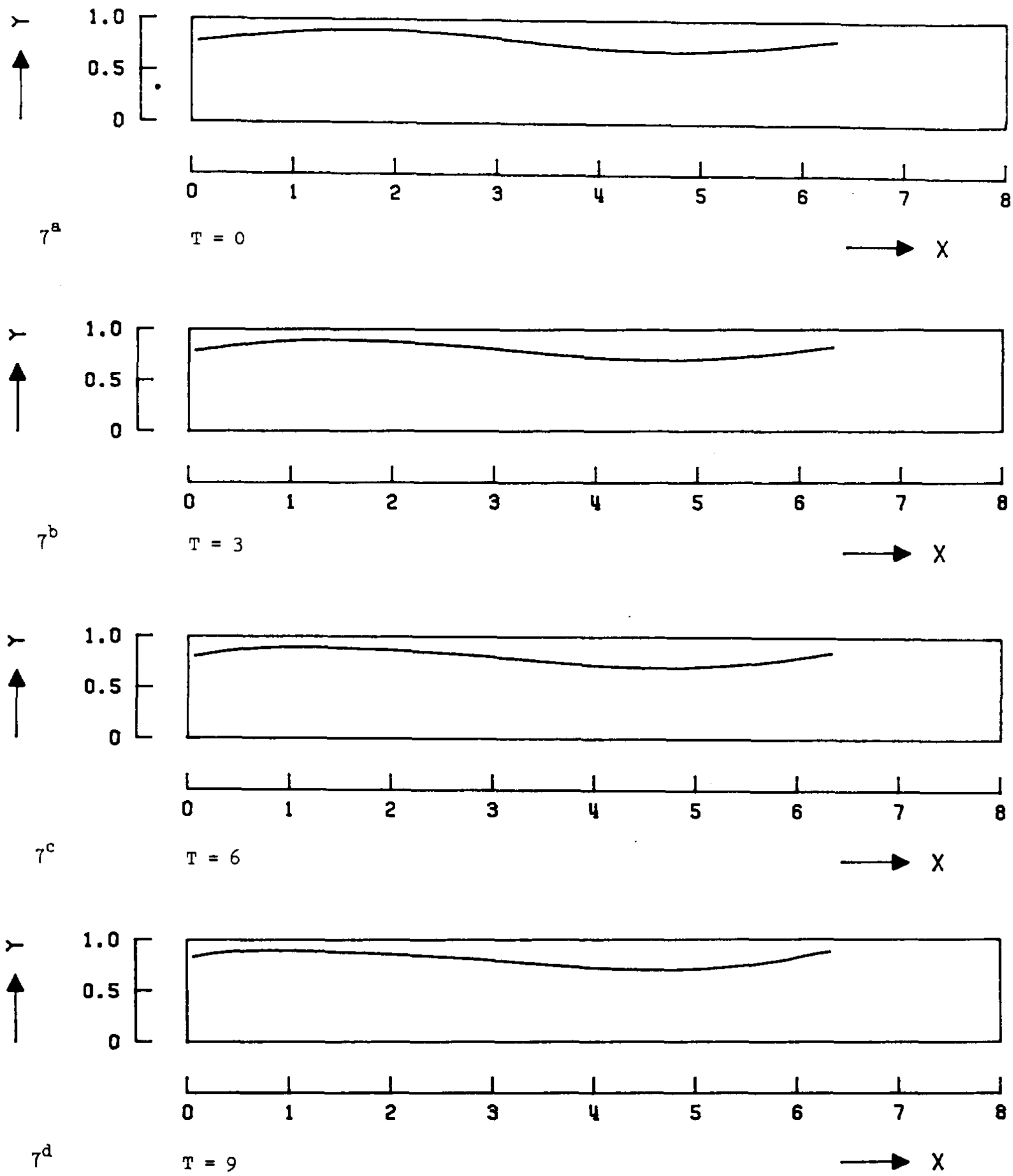

Figure 7. Development of interfacial wave with time when a counteractive pressure gradient is imposed. $C_1 = 10^{-4}$; $C_2 = 0$; $C_3 = -0.1$; $\overline{H}_1 = 0.8$; $A^{(0)} = 0.1$; $L = 6$; $\Delta T = 0.12$; $\Delta X = 0.066$.

shape of the wave is a funciton of the viscosity ratio of the fluids, the density difference between the fluids and the applied pressure gradient. For very small wave amplitudes, the shape of the wave does not change with time and its velocity agrees with that found by Yih in his first approximation.

The shape and amplitude of the calculated waves are in qualitative agreement with observations on oil–water core-annular flow experiments as reported by Ooms *et al.* (1984). However, accurate measurements are needed to be able to verify the reliability of our calculations in detail. At the moment, such experiments are being carried out.

The development of a sawtoothlike interface in some of the calculated cases suggests regions of high interfacial curvature. By neglecting the surface tension terms, the effect of capillary pressure gradients in these regions is not considered. In our future work, we hope to incorporate this effect in our calculations.

Acknowledgement—The authors are indebted to Prof.Dr.Ir. P. Wesseling of Delft University of Technology for stimulating discussions.

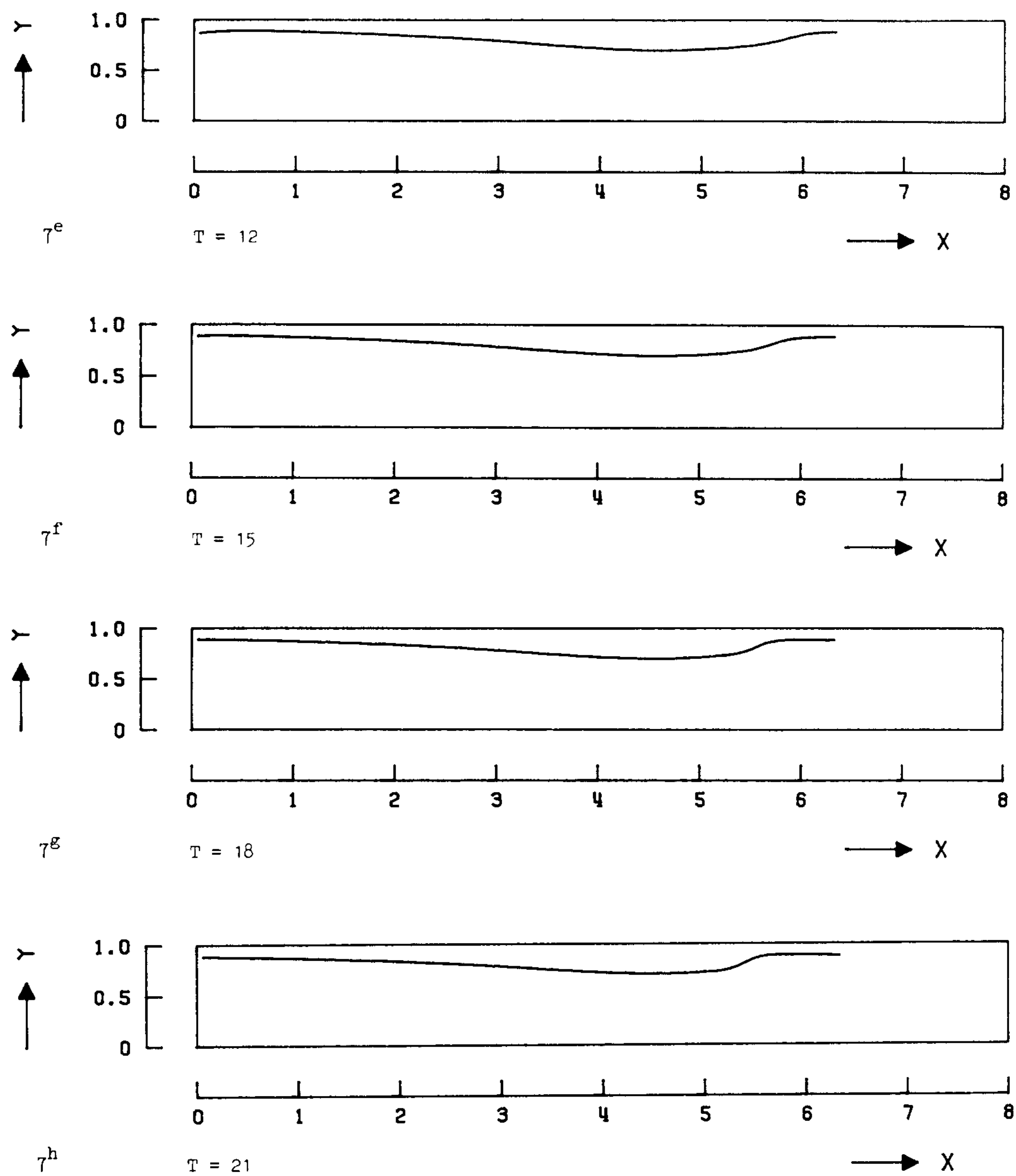

Figure 7. Continued.

REFERENCES

HICKOX, C. E. 1971 Stability of two fluids in a pipe. *Phys. Fluids* **14,** 251–262.

HOOPER, A. P. & BOYD, W. G. C. 1983 Shear-flow instability at the interface between two viscous fluids. *J. Fluid Mech.* **128,** 507–528.

LI, C. H. 1969 Instability of 3-layer viscous stratified fluid. *Phys. Fluids* **12,** 2473–2481.

OOMS, G., SEGAL, A., VAN DER WEES, A. J., MEERHOFF, R. & OLIEMANS, R. V. A. 1984 A theoretical model for core-annular flow of a very viscous oil core and a water annulus through a horizontal pipe. *Int. J. Multiphase Flow* **10,** 41–60.

YIH, C. S. 1967 Instability due to viscous stratification. *J. Fluid Mech.* **27,** 337–352.